Closed trajectories on symmetric convex Hamiltonian energy surfaces

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Abstract

In this article, let $\Sigma \subset \mathbf{R}^{2n}$ be a compact convex Hamiltonian energy surface which is symmetric with respect to the origin. where $n \geq 2$. We prove that there exist at least two geometrically distinct symmetric closed trajectories of the Reeb vector field on Σ.

Key words: Compact convex hypersurfaces, closed characteristics, Hamiltonian systems.

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1 Introduction and main results

In this article, let Σ be a fixed C^3 compact convex hypersurface in $\mathbf{R}^{2n}$, i.e., Σ is the boundary of a compact and strictly convex region U in $\mathbf{R}^{2n}$. We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose U contains the origin. We denote the set of all compact convex hypersurfaces which are symmetric with respect to the origin by $\mathcal{SH}(2n)$, i.e., $\Sigma = -\Sigma$ for $\Sigma \in \mathcal{SH}(2n)$. We consider closed characteristics (τ, y) on Σ, which are solutions of the following problem

$$\begin{cases} \dot{y} = JN_\Sigma(y), \\ y(\tau) = y(0), \end{cases} \tag{1.1}$$

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where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the identity matrix in $\mathbf{R}^n$, $\tau > 0$ and $N_\Sigma(y)$ is the outward normal vector of Σ at y normalized by the condition $N_\Sigma(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard inner product of $a, b \in \mathbf{R}^{2n}$. A closed characteristic (τ, y) is *prime* if τ is the minimal period of y. Two closed characteristics (τ, y) and (σ, z) are *geometrically distinct* if $y(\mathbf{R}) \neq z(\mathbf{R})$. We denote by $\mathcal{T}(\Sigma)$ the set of geometrically distinct closed characteristics (τ, y) on Σ. A closed characteristic (τ, y) on $\Sigma \in \mathcal{SH}(2n)$ is *symmetric* if $\{y(\mathbf{R})\} = \{-y(\mathbf{R})\}$, *non-symmetric* if $\{y(\mathbf{R})\} \cap \{-y(\mathbf{R})\} = \emptyset$. It was proved in [LLZ] that a prime characteristic (τ, y) on $\Sigma \in \mathcal{SH}(2n)$ is symmetric if and only if $y(t) = -y(t + \frac{\tau}{2})$ for all $t \in \mathbf{R}$.

There is a long standing conjecture on the number of closed characteristics on compact convex hypersurfaces in $\mathbf{R}^{2n}$:

$$^{\#}\mathcal{T}(\Sigma) \geq n, \qquad \forall\, \Sigma \in \mathcal{H}(2n). \tag{1.2}$$

Since the pioneering works [Rab1] of P. Rabinowitz and [Wei1] of A. Weinstein in 1978 on the existence of at least one closed characteristic on every hypersurface in $\mathcal{H}(2n)$, the existence of multiple closed characteristics on $\Sigma \in \mathcal{H}(2n)$ has been deeply studied by many mathematicians. When $n \geq 2$, besides many results under pinching conditions, in 1987-1988 I. Ekeland-L. Lassoued, I. Ekeland-H. Hofer, and A, Szulkin (cf. [EkL1], [EkH1], [Szu1]) proved

$$^{\#}\mathcal{T}(\Sigma) \geq 2, \qquad \forall\, \Sigma \in \mathcal{H}(2n).$$

In [HWZ] of 1998, H. Hofer-K. Wysocki-E. Zehnder proved that $^{\#}\mathcal{T}(\Sigma) = 2$ or ∞ holds for every $\Sigma \in \mathcal{H}(4)$. In [LoZ1] of 2002, Y. Long and C. Zhu proved

$$^{\#}\mathcal{T}(\Sigma) \geq [\frac{n}{2}] + 1, \qquad \forall\, \Sigma \in \mathcal{H}(2n),$$

where we denote by $[a] \equiv \max\{k \in \mathbf{Z} \,|\, k \leq a\}$. In [WHL], the authors proved the conjecture for $n = 3$. In [LLZ], the the authors proved the conjecture for $\Sigma \in \mathcal{SH}(2n)$.

Note that in [W2], the author proved if $^{\#}\mathcal{T}(\Sigma) = n$ for some $\Sigma \in \mathcal{SH}(2n)$ and $n = 2$ or 3, then any $(\tau, y) \in \mathcal{T}(\Sigma)$ is symmetric. Thus it is natural to conjecture that

$$^{\#}\mathcal{T}_s(\Sigma) \geq n, \qquad \forall\, \Sigma \in \mathcal{SH}(2n), \tag{1.3}$$

where $T_s(\Sigma)$ denotes the set of geometrically distinct symmetric closed characteristics (τ, y) on Σ.

The following is the main result in this article:

Theorem 1.1. *We have* $^{\#}\mathcal{T}_s(\Sigma) \geq 2$ *for any* $\Sigma \in \mathcal{SH}(2n)$*, where* $n \geq 2$.

In this article, let $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. Denote by

$a \cdot b$ and $|a|$ the standard inner product and norm in $\mathbf{R}^{2n}$. Denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the standard L^2-inner product and L^2-norm. For an S^1-space X, we denote by X_{S^1} the homotopy quotient of X module the S^1-action, i.e., $X_{S^1} = S^\infty \times_{S^1} X$. We define the functions

$$\begin{cases} [a] = \max\{k \in \mathbf{Z} \,|\, k \leq a\}, \quad E(a) = \min\{k \in \mathbf{Z} \,|\, k \geq a\}, \\ \qquad \varphi(a) = E(a) - [a], \end{cases} \tag{1.4}$$

Specially, $\varphi(a) = 0$ if $a \in \mathbf{Z}$, and $\varphi(a) = 1$ if $a \notin \mathbf{Z}$. In this article we use only $\mathbf{Q}$-coefficients for all homological modules.

2 A variational structure for closed characteristics

In this section, we transform the problem (1.1) into a fixed period problem of a Hamiltonian system and then study its variational structure.

In the rest of this paper, we fix a $\Sigma \in \mathcal{SH}(2n)$ and assume the following condition on Σ:

(F) **There exist only finitely many geometrically distinct symmetric closed characteristics $\{(\tau_j, y_j)\}_{1\leq j\leq k}$ on Σ.**

Note that $(\tau, y) \in \mathcal{T}_s(\Sigma)$ is a solution of (1.1) if and only if it satisfies the equation

$$\begin{cases} \dot{y} = JN_\Sigma(y), \\ y(\frac{\tau}{2}) = -y(0), \end{cases} \tag{2.1}$$

Now we construct a variational structure of closed characteristics as the following.

lemma 2.1. (cf. Proposition 2.2 of [WHL]) *For any sufficiently small $\vartheta \in (0,1)$, there exists a function $\varphi \equiv \varphi_\vartheta \in C^\infty(\mathbf{R}, \mathbf{R}^+)$ depending on ϑ which has 0 as its unique critical point in $[0, +\infty)$ such that the following hold*

(i) $\varphi(0) = 0 = \varphi'(0)$ and $\varphi''(0) = 1 = \lim_{t\to 0^+} \frac{\varphi'(t)}{t}$.

(ii) $\varphi(t)$ is a polynomial of degree 2 in a neighborhood of $+\infty$.

(iii) $\frac{d}{dt}\left(\frac{\varphi'(t)}{t}\right) < 0$ for $t > 0$, and $\lim_{t\to+\infty} \frac{\varphi'(t)}{t} < \vartheta$, i.e., $\frac{\varphi'(t)}{t}$ is strictly decreasing for $t > 0$.

(iv) $\min(\frac{\varphi'(t)}{t}, \varphi''(t)) \geq \sigma$ for all $t \in \mathbf{R}^+$ and some $\sigma > 0$. Consequently, φ is strictly convex on $[0, +\infty)$.

(v) In particular, we can choose $\alpha \in (1,2)$ sufficiently close to 2 and $c \in (0,1)$ such that $\varphi(t) = ct^\alpha$ whenever $\frac{\varphi'(t)}{t} \in [\vartheta, 1-\vartheta]$ and $t > 0$.

Let $j : \mathbf{R}^{2n} \to \mathbf{R}$ be the gauge function of Σ, i.e., $j(\lambda x) = \lambda$ for $x \in \Sigma$ and $\lambda \geq 0$, then $j \in C^3(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^0(\mathbf{R}^{2n}, \mathbf{R})$ and $\Sigma = j^{-1}(1)$. Denote by $\hat{\tau} = \inf_{1\leq j\leq k} \tau_j$ and $\hat{\sigma} = \min\{|y|^2 \,|\, y \in \Sigma\}$.

By the same proof of Proposition 2.4 of [WHL], we have the following

Proposition 2.2. *Let $a > \hat{\tau}$, $\vartheta_a \in \left(0, \frac{1}{a}\min\{\hat{\tau}, \hat{\sigma}\}\right)$ and φ_a be a C^∞ function associated to ϑ_a satisfying (i)-(iv) of Lemma 2.1. Define the Hamiltonian function $H_a(x) = a\varphi_a(j(x))$ and consider the fixed period problem*

$$\begin{cases} \dot{x}(t) = JH_a'(x(t)) \\ \quad x(\frac{1}{2}) = -x(0) \end{cases} \tag{2.2}$$

Then the following hold:

(i) $H_a \in C^3(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^1(\mathbf{R}^{2n}, \mathbf{R})$ and there exist $R, r > 0$ such that

$$r|\xi|^2 \le H_a''(x)\xi \cdot \xi \le R|\xi|^2, \quad \forall x \in \mathbf{R}^{2n} \setminus \{0\},\ \xi \in \mathbf{R}^{2n}.$$

(ii) There exist $\epsilon_1, \epsilon_2 \in \left(0, \frac{1}{2}\right)$ and $C \in \mathbf{R}$, such that

$$\frac{\epsilon_1|x|^2}{2} - C \le H_a(x) \le \frac{\epsilon_2|x|^2}{2} + C, \quad \forall x \in \mathbf{R}^{2n}.$$

(iii) Solutions of (2.2) are $x \equiv 0$ and $x = \rho y(\tau t)$ with $\frac{\varphi_a'(\rho)}{\rho} = \frac{\tau}{a}$, where (τ, y) is a solution of (2.1). In particular, nonzero solutions of (2.2) are in one to one correspondence with solutions of (2.1) with period $\tau < a$.

(iv) There exists $r_0 > 0$ independent of a and there exists $\mu_a > 0$ depending on a such that

$$H_a''(x)\xi \cdot \xi \ge 2ar_0|\xi|^2, \qquad \text{for} \quad 0 < |x| \le \mu_a,\ \xi \in \mathbf{R}^{2n}.$$

In the following, we will use the Clarke-Ekeland dual action principle. As usual, the Fenchel transform of a function $F : \mathbf{R}^{2n} \to \mathbf{R}$ is defined by

$$F^*(y) = \sup\{x \cdot y - F(x) \mid x \in \mathbf{R}^{2n}\}. \tag{2.3}$$

Following Proposition 2.2.10 of [Eke3], Lemma 3.1 of [Eke1] and the fact that $F_1 \le F_2 \Leftrightarrow F_1^* \ge F_2^*$, we have:

Proposition 2.3. *Let H_a be a function defined in Proposition 2.2 and $G_a = H_a^*$ the Fenchel transform of H_a. Then we have*

(i) $G_a \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^1(\mathbf{R}^{2n}, \mathbf{R})$ and

$$G_a'(y) = x \Leftrightarrow y = H_a'(x) \Rightarrow H_a''(x)G_a''(y) = 1.$$

(ii) G_a is strictly convex. Let R and r be the real numbers given by (i) of Proposition 2.2. Then we have

$$R^{-1}|\xi|^2 \le G_a''(y)\xi \cdot \xi \le r^{-1}|\xi|^2, \quad \forall y \in \mathbf{R}^{2n} \setminus \{0\},\ \xi \in \mathbf{R}^{2n}.$$

(iii) Let $\epsilon_1, \epsilon_2, C$ be the real numbers given by (ii) of Proposition 2.2. Then we have

$$\frac{|x|^2}{2\epsilon_2} - C \leq G_a(x) \leq \frac{|x|^2}{2\epsilon_1} + C, \quad \forall x \in \mathbf{R}^{2n}.$$

(iv) Let $r_0 > 0$ be the constant given by (iv) of Proposition 2.2. Then there exists $\eta_a > 0$ depending on a such that the following holds

$$G_a''(y)\xi \cdot \xi \leq \frac{1}{2ar_0}|\xi|^2, \qquad \text{for} \quad 0 < |y| \leq \eta_a, \ \xi \in \mathbf{R}^{2n}.$$

(v) In particular, let $H_a = a\varphi_a(j(x))$ with φ_a satisfying further (v) of Lemma 2.1. Then we have $G_a(\mu j'(z)) = c_1\mu^\beta$ when $z \in \Sigma$ and $\mu j'(z) \in \{H_a'(x) \mid H_a(x) = acj(x)^\alpha\}$, where c is given by (v) of Lemma 2.1, $c_1 > 0$ is some constant depending on a and $\alpha^{-1} + \beta^{-1} = 1$. ∎

Now we apply the dual action principle to problem (2.3). Let

$$L^2\left(\mathbf{R} \Big/ \left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right) = \{u \in L^2(\mathbf{R}, \mathbf{R}^{2n}) | u(t+1/2) = -u(t)\}. \tag{2.4}$$

Define a linear operator $M : L^2\left(\mathbf{R}/\left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right) \to L^2\left(\mathbf{R}/\left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right)$ by

$$\frac{d}{dt}Mu(t) = u(t). \tag{2.5}$$

Lemma 2.4. *M is a compact operator from $L^2\left(\mathbf{R}/\left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right)$ into itself and $M^* = -M$.*

Proof. Note that M sends $L^2\left(\mathbf{R}/\left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right)$ into $W^{1,2}([0,1/2], \mathbf{R}^{2n})$, and the identity map from $W^{1,2}([0,1/2], \mathbf{R}^{2n})$ to $L^2\left(\mathbf{R}/\left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right)$ is compact by the Rellich-Kondrachov theorem. Hence M is compact.

To check it is anti-symmetric, we use integrate by parts:

$$\int_0^{1/2}(Mu, v)dt = -\int_0^{1/2}(u, Mv)dt + (Mu, Mv)|_0^{1/2}.$$

and the last term vanishes since $Mu(1/2) = -Mu(0)$ and $Mv(1/2) = -Mv(0)$. Hence M is anti-symmetric. ∎

The dual action functional on $L^2\left(\mathbf{R}/\left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right)$ is defined by

$$\Psi_a(u) = \int_0^{1/2}\left(\frac{1}{2}Ju \cdot Mu + G_a(-Ju)\right)dt, \tag{2.6}$$

where G_a is given by Proposition 2.3.

Proposition 2.5. *The functional Ψ_a is bounded from below on $L^2\left(\mathbf{R}/\left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right)$.*

Proof. For any $u \in L^2\left(\mathbf{R}/\left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right)$, we represent u by its Fourier series

$$u(t) = \sum_{k \in 2\mathbf{Z}+1} e^{2k\pi Jt}x_k, \quad x_k \in \mathbf{R}^{2n}. \tag{2.7}$$

Then we have

$$Mu(t) = -J \sum_{k\in 2\mathbf{Z}+1} \frac{1}{2\pi k} e^{2k\pi J t} x_k. \tag{2.8}$$

Hence

$$\frac{1}{2}\langle Ju,\ Mu\rangle = -\frac{1}{2} \sum_{k\in 2\mathbf{Z}+1} \frac{1}{2\pi k}|x_k|^2 \geq -\frac{1}{4\pi}\|u\|^2. \tag{2.9}$$

By (2.6), we have

$$\begin{aligned}
\Psi_a(u) &= \int_0^{1/2} \left(\frac{1}{2}Ju\cdot Mu + G_a(-Ju)\right) dt \\
&\geq \frac{1}{2}\langle Ju,\ Mu\rangle + \int_0^{1/2} \left(\frac{|u|^2}{2\epsilon_2} - C\right) dt. \\
&\geq \left(\frac{1}{2\epsilon_2} - \frac{1}{4\pi}\right) \|u\|^2 - C \\
&\geq C_1\|u\|^2 - C
\end{aligned} \tag{2.10}$$

for some constant $C_1 > 0$, where in the first inequality, we have used (iii) of Proposition 2.3. Hence the proposition holds. ∎

Proposition 2.6. *The functional Ψ_a is $C^{1,1}$ on $L^2\left(\mathbf{R}/\ \left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right)$ and satisfies the Palais-Smale condition. Suppose x is a solution of (2.2), then $u = \dot{x}$ is a critical point of Ψ_a. Conversely, suppose u is a critical point of Ψ_a, then Mu is a solution of (2.2). In particular, solutions of (2.2) are in one to one correspondence with critical points of Ψ_a.*

Proof. By (ii) of Proposition 2.3 and the same proof of Proposition 3.3 on p.33 of [Eke1], we have Ψ_a is $C^{1,1}$ on $L^2\left(\mathbf{R}/\ \left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right)$. By (2.10) and the proof of Lemma 5.2.8 of [Eke3], we have Ψ_a satisfies the Palais-Smale condition.

By (2.6), we have

$$\langle \Psi_a'(u), v\rangle = \langle Mu, Jv\rangle - \langle G_a'(-Ju), Jv\rangle, \tag{2.11}$$

where we use the fact that

$$Mu(t) = \int_0^t u(s)ds - \frac{1}{2}\int_0^{1/2} u(s)ds, \tag{2.12}$$

and $MJu(t) = JMu(t)$. Hence $\Psi_a'(u) = 0$ if and only if $Mu = G_a'(-Ju)$, where we used the fact $G_a'(-Ju(\frac{1}{2})) = G_a'(Ju(0)) = -G_a'(-Ju(0))$. Taking Frenchel dual we have $-Ju = H_a'(Mu)$, i.e., $u = JH_a'(Mu)$. Hence Mu is a solution of (2.2). The converse is obvious. ∎

Proposition 2.7. *We have $\Psi_a(u_a) < 0$ for every critical point $u_a \neq 0$ of Ψ_a.*

Proof. By Propositions 2.2 and 2.6, we have $u_a = \dot{x}_a$ and $x_a = \rho_a y(\tau t)$ with

$$\frac{\varphi_a'(\rho_a)}{\rho_a} = \frac{\tau}{a}. \tag{2.13}$$

Hence we have

$$\begin{aligned}\Psi_a(u_a) &= \int_0^{1/2} \left(\frac{1}{2} J\dot{x}_a \cdot x_a + G_a(-J\dot{x}_a)\right) dt \\ &= -\frac{1}{4}\langle H_a'(x_a),\ x_a\rangle + \int_0^{1/2} G_a(H_a'(x_a))dt \\ &= \frac{1}{4} a\varphi_a'(\rho_a)\rho_a - \frac{1}{2} a\varphi_a(\rho_a). \end{aligned} \tag{2.14}$$

Here the second equality follows from (2.2) and the third equality follows from (i) of Proposition 2.3 and (2.3).

Let $f(t) = \frac{1}{2}a\varphi_a'(t)t - a\varphi_a(t)$ for $t \geq 0$. Then we have $f(0) = 0$ and $f'(t) = \frac{a}{2}(\varphi_a''(t)t - \varphi_a'(t)) < 0$ since $\frac{d}{dt}(\frac{\varphi_a'(t)}{t}) < 0$ by (iii) of Lemma 2.1. This together with (2.13) yield the proposition. ∎

We have a natural S^1-action on $L^2\left(\mathbf{R}/\ \left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right)$ defined by

$$\theta * u(t) = u(\theta + t), \quad \forall \theta \in S^1 \equiv \mathbf{R}/\mathbf{Z}, t \in \mathbf{R}. \tag{2.15}$$

Then we have

Lemma 2.8. *The functional Ψ_a is S^1-invariant.*

Proof. Note that we have the following

Claim. *We have $M(\theta * u) = \theta * (Mu)$.*

In fact, by (2.12), we have

$$\begin{aligned} M(\theta * u)(t) &= \int_0^t \theta * u(s)ds - \frac{1}{2}\int_0^{1/2} \theta * u(s)ds \\ &= \int_0^t u(\theta + s)ds - \frac{1}{2}\int_0^{1/2} u(\theta + s)ds \\ &= \int_\theta^{t+\theta} u(s)ds - \frac{1}{2}\int_\theta^{1/2+\theta} \cdot u(s)ds \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \theta * (Mu)(t) &= \theta * \left(\int_0^t u(s)ds - \frac{1}{2}\int_0^{1/2} u(s)ds\right) \\ &= \int_0^{t+\theta} u(s)ds - \frac{1}{2}\int_0^{1/2} u(s)ds \\ &= \int_0^\theta u(s)ds + \int_\theta^{t+\theta} u(s)ds - \frac{1}{2}\int_0^\theta u(s)ds - \frac{1}{2}\int_\theta^{1/2} u(s)ds \\ &= \frac{1}{2}\int_0^\theta u(s)ds + \int_\theta^{t+\theta} u(s)ds - \frac{1}{2}\int_\theta^{1/2} u(s)ds \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_{1/2}^{\theta+1/2} u(s)ds + \int_{\theta}^{t+\theta} u(s)ds - \frac{1}{2}\int_{\theta}^{1/2} u(s)ds && (2.16)\\
&= \int_{\theta}^{t+\theta} u(s)ds - \frac{1}{2}\int_{\theta}^{1/2+\theta} \cdot u(s)ds,
\end{aligned}
$$

where in (2.16), we use the fact $u(t+1/2) = -u(t)$. Hence the claim holds.

Now we have

$$
\begin{aligned}
\Psi_a(\theta * u) &= \int_0^{1/2} \left(\frac{1}{2}J(\theta * u)\cdot M(\theta * u) + G_a(-J(\theta * u))\right) dt,\\
&= \int_0^{1/2} \left(\frac{1}{2}\theta * (Ju)\cdot \theta * (Mu) + G_a(\theta * (-Ju))\right) dt,\\
&= \int_{\theta}^{\theta+1/2} \left(\frac{1}{2}Ju\cdot Mu + G_a(-Ju)\right) dt\\
&= \int_{\theta}^{1/2} \left(\frac{1}{2}Ju\cdot Mu + G_a(-Ju)\right) dt + \int_{1/2}^{\theta+1/2} \left(\frac{1}{2}Ju\cdot Mu + G_a(-Ju)\right) dt\\
&= \int_{\theta}^{1/2} \left(\frac{1}{2}Ju\cdot Mu + G_a(-Ju)\right) dt + \int_{0}^{\theta} \left(\frac{1}{2}(-Ju)\cdot (-Mu) + G_a(Ju)\right) dt\\
&= \int_{\theta}^{1/2} \left(\frac{1}{2}Ju\cdot Mu + G_a(-Ju)\right) dt = \Psi_a(u),
\end{aligned}
$$

where in the above computation, we use $u(t+1/2) = -u(t)$ and $G_a(x) = G_a(-x)$, which follows from $\Sigma = -\Sigma$. Hence the proposition holds. ∎

For any $\kappa \in \mathbf{R}$, we denote by

$$
\Lambda_a^{\kappa} = \left\{ u \in L^2\left(\mathbf{R}\Big/ \left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right) \;\Big|\; \Psi_a(u) \le \kappa \right\}. \tag{2.17}
$$

For a critical point u of Ψ_a, we denote by

$$
\Lambda_a(u) = \Lambda_a^{\Psi_a(u)} = \left\{ w \in L^2\left(\mathbf{R}\Big/ \left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right) \;\Big|\; \Psi_a(w) \le \Psi_a(u) \right\}. \tag{2.18}
$$

Clearly, both sets are S^1-invariant. Since the S^1-action preserves Ψ_a, if u is a critical point of Ψ_a, then the whole orbit $S^1\cdot u$ is formed by critical points of Ψ_a. Denote by $crit(\Psi_a)$ the set of critical points of Ψ_a. Note that by the condition (F), (iii) of Proposition 2.2 and Proposition 2.6, the number of critical orbits of Ψ_a is finite. Hence as usual we can make the following definition.

Definition 2.9. *Suppose u is a nonzero critical point of Ψ_a, and $\mathcal{N}$ is an S^1-invariant open neighborhood of $S^1\cdot u$ such that $crit(\Psi_a)\cap(\Lambda_a(u)\cap\mathcal{N}) = S^1\cdot u$. Then the S^1-critical modules of $S^1\cdot u$ is defined by*

$$
\begin{aligned}
C_{S^1,\,q}(\Psi_a,\ S^1\cdot u) &= H_{S^1,\,q}(\Lambda_a(u)\cap\mathcal{N},\ (\Lambda_a(u)\setminus S^1\cdot u)\cap\mathcal{N})\\
&\equiv H_q((\Lambda_a(u)\cap\mathcal{N})_{S^1},\ ((\Lambda_a(u)\setminus S^1\cdot u)\cap\mathcal{N})_{S^1}), && (2.19)
\end{aligned}
$$

where $H_{S^1,\,*}$ is the S^1-equivariant homology with rational coefficients in the sense of A. Borel (cf. Chapter IV of [Bor1]).

By the same argument as Proposition 3.2 of [WHL], we have the following proposition for critical modules.

Proposition 2.10. *The critical module $C_{S^1,\,q}(\Psi_a,\ S^1\cdot u)$ is independent of the choice of H_a defined in Proposition 2.2 in the sense that if x_i are solutions of (2.2) with Hamiltonian functions $H_{a_i}(x)\equiv a_i\varphi_{a_i}(j(x))$ for $i=1$ and 2 respectively such that both x_1 and x_2 correspond to the same closed characteristic (τ, y) on Σ. Then we have*

$$C_{S^1,\,q}(\Psi_{a_1},\ S^1\cdot\dot{x}_1)\cong C_{S^1,\,q}(\Psi_{a_2},\ S^1\cdot\dot{x}_2),\quad \forall q\in\mathbf{Z}. \tag{2.20}$$

In other words, the critical modules are invariant for all $a>\tau$ and φ_a satisfying (i)-(iv) of Lemma 2.1.

In order to compute the critical modules, as in p.35 of [Eke1] and p.219 of [Eke3] we introduce the following.

Definition 2.11. *Suppose u is a nonzero critical point of Ψ_a. Then the formal Hessian of Ψ_a at u is defined by*

$$Q_a(v,\ v)=\int_0^{1/2}(Jv\cdot Mv+G_a''(-Ju)Jv\cdot Jv)dt, \tag{2.21}$$

which defines an orthogonal splitting $L^2\left(\mathbf{R}/\ \left(\frac{1}{2}\mathbf{Z}\right),\mathbf{R}^{2n}\right)=E_-\oplus E_0\oplus E_+$ of $L^2\left(\mathbf{R}/\ \left(\frac{1}{2}\mathbf{Z}\right),\mathbf{R}^{2n}\right)$ into negative, zero and positive subspaces. The index of u is defined by $i(u)=\dim E_-$ and the nullity of u is defined by $\nu(u)=\dim E_0$.

Next we show that the index and nullity defined as above are the Morse index and nullity of a corresponding functional on a finite dimensional subspace of $L^2\left(\mathbf{R}/\ \left(\frac{1}{2}\mathbf{Z}\right),\mathbf{R}^{2n}\right)$.

Lemma 2.12. *Let Ψ_a be a functionals defined by (2.6). Then there exists a finite dimensional S^1-invariant subspace X of $L^2\left(\mathbf{R}/\ \left(\frac{1}{2}\mathbf{Z}\right),\mathbf{R}^{2n}\right)$ and a S^1-equivariant map $h_a: X\to X^{\perp}$ such that the following hold*

(i) For $g\in X$, each function $h\mapsto\Psi_a(g+h)$ has $h_a(g)$ as the unique minimum in $X^{\perp}$.

Let $\psi_a(g)=\Psi_a(g+h_a(g))$. Then we have

(ii) The function ψ_a is C^1 on X and S^1-invariant. g_a is a critical point of ψ_a if and only if $g_a+h_a(g_a)$ is a critical point of Ψ_a.

(iii) If $g_a\in X$ and H_a is C^k with $k\geq 2$ in a neighborhood of the trajectory of $g_a+h_a(g_a)$, then ψ_a is C^{k-1} in a neighborhood of g_a. In particular, if g_a is a nonzero critical point of ψ_a, then ψ_a is C^2 in a neighborhood of the critical orbit $S^1\cdot g_a$. The index and nullity of Ψ_a at $g_a+h_a(g_a)$ defined in Definition 2.11 coincide with the Morse index and nullity of ψ_a at g_a.

(iv) For any $\kappa \in \mathbf{R}$, we denote by

$$\widetilde{\Lambda}_a^\kappa = \{g \in X \mid \psi_a(g) \le \kappa\}. \tag{2.22}$$

Then the natural embedding $\widetilde{\Lambda}_a^\kappa \hookrightarrow \Lambda_a^\kappa$ given by $g \mapsto g + h_a(g)$ is an S^1-equivariant homotopy equivalence.

Proof. By (ii) of Proposition 2.3, we have

$$(G_a'(u) - G_a'(v), u - v) \ge \omega |u - v|^2, \quad \forall u, v \in \mathbf{R}^{2n}, \tag{2.23}$$

for some $\omega > 0$. Hence we can use the proof of Proposition 3.9 of [Vit1] to obtain X and h_a. In fact, X is the subspace of $L^2\left(\mathbf{R}/\left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right)$ generated by the eigenvectors of $-JM$ whose eigenvalues are less than $-\frac{\omega}{2}$ and $h_a(g)$ is defined by the equation

$$\frac{\partial}{\partial h}\Psi_a(g + h_a(g)) = 0,$$

then (i)-(iii) follows from Proposition 3.9 of [Vit1]. (iv) follows from Lemma 5.1 of [Vit1]. ∎

Note that Ψ_a is not C^2 in general, and then we can not apply Morse theory to Ψ_a directly. After the finite dimensional approximation, the function ψ_a has much better differentiability, which allows us to apply the Morse theory to study its property.

Proposition 2.13. *Let Ψ_a be a functional defined by (2.6), and $u_a = \dot{x}_a$ be the critical point of Ψ_a so that x_a corresponds to a closed characteristic (τ, y) on Σ. Then the nullity $\nu(u_a)$ of the functional Ψ_a at its critical point u_a is the number of linearly independent solutions of the boundary value problem*

$$\begin{cases} \dot{\xi}(t) = JH_a''(x_a(t))\xi \\ \quad \xi(\frac{1}{2}) = -\xi(0) \end{cases} \tag{2.24}$$

Proof. By (2.21), we have

$$\begin{aligned} Q_a(v,\ w) &= \int_0^{1/2} (Jv \cdot Mw + G_a''(-Ju)Jv \cdot Jw)dt, \\ &= \langle Mw, Jv\rangle + \langle (H_a''(x_a(t))^{-1}Jw, Jv\rangle \end{aligned} \tag{2.25}$$

where we have used (2.2) and (i) of Proposition 2.3. Now $w \in E_0$ if and only if $Q_a(v,\ w) = 0$ for any $v \in L^2\left(\mathbf{R}/\left(\frac{1}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right)$. Hence we must have $Mw + (H_a''(x_a(t))^{-1}Jw = 0$, i.e., we have $w = JH_a''(x_a(t))Mw$. Hence Mw solves (2.25). ∎

Denote by $R(t)$ the fundamental solution of the linearized system

$$\dot{\xi}(t) = JH_a''(x_a(t))\xi(t), \tag{2.26}$$

Then we have the following

Proposition 2.14. *In an appropriate coordinates there holds*

$$R(1/2)=\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{with} \quad A=\begin{pmatrix} -1 & -\gamma \\ 0 & -1 \end{pmatrix},$$

with $\gamma>0$ and C is independent of H_a.

Proof. Note that by Lemma 1.6.11 of [Eke3], we have

$$R(t)T_{y(0)}\Sigma \subset T_{y(\tau t)}\Sigma. \tag{2.27}$$

Differentiating (2.2) and use the fact $x_a(t+1/2)=-x_a(t)$, we have

$$R(1/2)\dot{x}_a(0)=-\dot{x}_a(0). \tag{2.28}$$

Let

$$x_a(\rho,t)=\rho y\left(\frac{\tau t}{T_\rho}\right) \quad \text{with } \frac{\tau}{T_\rho}=\frac{a\varphi_a'(\rho)}{\rho}. \tag{2.29}$$

Then we have $x_a(\rho,T_\rho/2)=-x_a(\rho,0)$. Differentiating it with respect to ρ and using (2.29) together with $\dot{x}_a(1/2)=-\dot{x}_a(0)$, we get

$$-\frac{\tau}{2a}\frac{d}{d\rho}\left(\frac{\rho}{\varphi_a'(\rho)}\right)\dot{x}_a(0)+R(1/2)\rho^{-1}x_a(0)=-\rho^{-1}x_a(0).$$

Hence we have

$$R(1/2)x_a(0)=-x_a(0)+\frac{\rho\tau}{2a}\frac{d}{d\rho}\left(\frac{\rho}{\varphi_a'(\rho)}\right)\dot{x}_a(0)=x_a(0)+\gamma\dot{x}_a(0), \tag{2.30}$$

where $\gamma>0$ since $\frac{d}{d\rho}\left(\frac{\rho}{\varphi_a'(\rho)}\right)>0$ by (iii) of Proposition 2.1. For any $w\in\mathbf{R}^{2n}$, we have

$$\begin{aligned} H_a''(x_a)w &= a\varphi_a''(j(x_a))(j'(x_a),w)j'(x_a)+a\varphi_a'(j(x_a))j''(x_a)w \\ &= a\varphi_a''(j(x_a))(j'(y),w)j'(y)+\tau j''(y)w. \end{aligned} \tag{2.31}$$

The last equality follows from (iii) of Proposition 2.2. Let $z(t)=R(t)z(0)$ for $z(0)\in T_{y(0)}\Sigma$. Then by (2.27), we have $\dot{z}(t)=\tau j''(y(t))z(t)$. Therefore $R(1/2)|_{T_{y(0)}}\Sigma$ is independent of the choice of H_a in Proposition 2.2. Summing up, we have proved that in an appropriate coordinates there holds

$$R(1/2)=\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{with} \quad A=\begin{pmatrix} -1 & -\gamma \\ 0 & -1 \end{pmatrix},$$

with C is independent of H_a, where we use $\{-\dot{x}_a(0),x_a(0),e_1,\ldots,e_{2n-2}\}$ as an basis of $\mathbf{R}^{2n}$. ∎

Proposition 2.15. *Let Ψ_a be a functional defined by (2.6), and u be a nonzero critical point of Ψ_a. Then we have*

$$C_{S^1,\,q}(\Psi_a,\ S^1\cdot u)=0, \qquad \forall q\notin[i(u),i(u)+\nu(u)-1]. \tag{2.32}$$

Proof. By (iv) of Lemma 2.12, we have

$$C_{S^1,\,q}(\Psi_a,\ S^1\cdot u) \simeq C_{S^1,\,q}(\psi_a,\ S^1\cdot u), \tag{2.33}$$

where $C_{S^1,\,q}(\psi_a,\ S^1\cdot u) = H_{S^1,\,q}(\widetilde{\Lambda}_a(u)\cap\mathcal{N},\ (\widetilde{\Lambda}_a(u)\setminus S^1\cdot u)\cap\mathcal{N})$ and $\mathcal{N}$ is an S^1-invariant open neighborhood of $S^1\cdot u$ such that $crit(\psi_a)\cap(\widetilde{\Lambda}_a(u)\cap\mathcal{N}) = S^1\cdot u$. By (iii) of Lemma 2.12, the functional ψ_a is C^2 near $S^1\cdot u$. Thus we can use the Gromoll-Meyer theory in the equivariant sense to obtain the proposition. ∎

Recall that for a principal $U(1)$-bundle $E\to B$, the Fadell-Rabinowitz index (cf. [FaR1]) of E is defined to be $\sup\{k\mid c_1(E)^{k-1}\neq 0\}$, where $c_1(E)\in H^2(B,\mathbf{Q})$ is the first rational Chern class. For a $U(1)$-space, i.e., a topological space X with a $U(1)$-action, the Fadell-Rabinowitz index is defined to be the index of the bundle $X\times S^\infty\to X\times_{U(1)}S^\infty$, where $S^\infty\to CP^\infty$ is the universal $U(1)$-bundle. For any $\kappa\in\mathbf{R}$, we denote by

$$\Psi_a^{\kappa-} = \left\{w\in L^2\left(\mathbf{R}\Big/\left(\frac{1}{2}\mathbf{Z}\right),\mathbf{R}^{2n}\right)\ \Big|\ \Psi_a(w)<\kappa\right\}. \tag{2.34}$$

Then as in P.218 of [Eke3], we define

$$c_i = \inf\{\delta\in\mathbf{R}\mid \hat{I}(\Psi_a^{\kappa-})\geq i\}, \tag{2.35}$$

where $\hat{I}$ is the Fadell-Rabinowitz index given above. Then as Proposition 3 in P.218 of [Eke3], we have

Proposition 2.16. *Every c_i is a critical value of Ψ_a. If $c_i = c_j$ for some $i<j$, then there are infinitely many geometrically distinct symmetric closed characteristics on Σ.*

By a similar argument as Proposition 3.5 of [W1] and Proposition 2.15, we have

Proposition 2.17. *Suppose u is the critical point of Ψ_a found in Proposition 2.16. Then we have*

$$\Psi_a(u) = c_i,\qquad C_{S^1,\,2(i-1)}(\Psi_a,\ S^1\cdot u)\neq 0. \tag{2.36}$$

In particular, we have $i(u)\leq 2(i-1)\leq i(u)+\nu(u)-1$.

3 Index iteration theory for symmetric closed characteristics

In this section, we study the index iteration theory for symmetric closed characteristics.

Note that if $(\tau,y)\in\mathcal{T}_s(\Sigma)$, then $((2m-1)\tau,y)$ is a solution of (2.1) for any $m\in\mathbf{N}$. Thus $((2m-1)\tau,y)$ corresponds to a critical point of Ψ_a via Propositions 2.2 and 2.6, we denote it by u^{2m-1}. First note that we have the following

Lemma 3.1. *Suppose u^{2m-1} is a nonzero critical point of Ψ_a such that u corresponds to $(\tau, y) \in \mathcal{T}_s(\Sigma)$. Let $H(x) = j(x)^2$, where j is the gauge function of Σ. Then $i(u^{2m-1})$ equals the index of the following quadratic form*

$$Q_{(2m-1)\tau/2}(\xi,\ \xi) = \int_0^{(2m-1)\tau/2} (J\dot{\xi} \cdot \xi + (H''(y(t)))^{-1} J\dot{\xi} \cdot J\dot{\xi}) dt, \tag{3.1}$$

where $\xi \in W^{1,2}\left(\mathbf{R}/\ \left(\frac{(2m-1)\tau}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right) \equiv \{w \in W^{1,2}(\mathbf{R}, \mathbf{R}^{2n}) | w(t + \frac{(2m-1)\tau}{2}) = -w(t)\}$.. Moreover, we have $\nu(u^{2m-1}) = \mathrm{nullity} Q_{(2m-1)\tau/2} - 1$.

Proof. By a similar argument as in proposition 1.7.5 and P.36 of [Eke3] and Proposition 3.5 of [WHL], we obtain the lemma. ∎

Suppose u^{2k-1} is a nonzero critical point of Ψ_a such that u corresponds to $(\tau, y) \in \mathcal{T}_s(\Sigma)$. Then for any $\omega \in \mathbf{U}$, let

$$Q^{\omega}_{(2k-1)\tau/2}(\xi,\ \xi) = \int_0^{(2k-1)\tau/2} (J\dot{\xi} \cdot \xi + (H''(y(t)))^{-1} J\dot{\xi} \cdot J\dot{\xi}) dt, \tag{3.2}$$

where $\xi \in E^{\omega}_{(2k-1)\tau/2} \equiv \{u \in W^{1,2}([0, (2k-1)\tau/2], \mathbf{C}^{2n}) | w(\frac{(2k-1)\tau}{2}) = \omega w(0)\}$..

Clearly the quadratic form $Q_{(2m-1)\tau/2}$ on the real Hilbert space $W^{1,2}\left(\mathbf{R}/\ \left(\frac{(2m-1)\tau}{2}\mathbf{Z}\right), \mathbf{R}^{2n}\right)$ and the Hermitian form $Q^{-1}_{(2m-1)\tau/2}$ on the complex Hilbert space $E^{-1}_{(2m-1)\tau/2}$ have the same index.

If $\omega^{2m-1} = -1$, we identify $E^{\omega}_{\tau/2}$ with a subspace of $E^{-1}_{(2m-1)\tau/2}$ via

$$E^{\omega}_{\tau/2} = \{u \in W^{1,2}(\mathbf{R}, \mathbf{C}^{2n}) | w(t + \tau/2) = \omega w(t)\}. \tag{3.3}$$

Note that if $\xi \in E^{\omega}_{\tau/2}$, we have

$$\begin{aligned} Q^{\omega}_{(2m-1)\tau/2}(\xi,\ \xi) &= \int_0^{(2m-1)\tau/2} (J\dot{\xi} \cdot \xi + (H''(y(t)))^{-1} J\dot{\xi} \cdot J\dot{\xi}) dt \\ &= \sum_{k=0}^{2m-1} (\omega\overline{\omega})^k \int_0^{\tau/2} (J\dot{\xi} \cdot \xi + (H''(y(t)))^{-1} J\dot{\xi} \cdot J\dot{\xi}) dt \\ &= (2m-1) Q^{\omega}_{\tau/2}(\xi, \xi). \end{aligned} \tag{3.4}$$

Lemma 3.2. *The spaces $E^{\omega}_{\tau/2}$ for $\omega^{2m-1} = -1$ are orthogonal subspaces of $E^{-1}_{(2m-1)\tau/2}$, both for the standard Hilbert structure and for $Q^{-1}_{(2m-1)\tau/2}$, and we have the decomposition*

$$E^{-1}_{(2m-1)\tau/2} = \bigoplus_{\omega^{2m-1}=-1} E^{\omega}_{\tau/2}. \tag{3.5}$$

Proof. Any $\xi \in E^{-1}_{(2m-1)\tau/2}$ can be written as

$$\xi(t) = \sum_{p \in 2\mathbf{Z}+1} x_p \exp\left(\frac{2i\pi pt}{(2m-1)\tau}\right) \tag{3.6}$$

for $q = 1, 3, \ldots, 4m-3$, denote by $C(q)$ the set of all p such that $p - q \in (4m-2)\mathbf{Z}$. Thus we may write

$$\xi(t) = \sum_{\substack{q\in 2\mathbf{Z}+1 \\ 1\le q\le 4m-3}} \xi_q(t), \qquad \xi_q(t) = \sum_{C(q)} x_p \exp\left(\frac{2i\pi pt}{(2m-1)\tau}\right) \tag{3.7}$$

Then we have

$$\begin{aligned} \xi_q(t+\tau/2) &= \sum_{C(q)} x_p \exp\left(\frac{2i\pi pt}{(2m-1)\tau} + \frac{i\pi p}{2m-1}\right) \\ &= \exp\left(\frac{i\pi q}{2m-1}\right)\xi_q(t). \end{aligned} \tag{3.8}$$

Thus $\xi_q \in E^{\omega}_{\tau/2}$ with $\omega = \exp\left(\frac{i\pi q}{2m-1}\right)$, when q runs from $1, 3, \ldots, 4m-3$, then ω runs through the $2m-1$ roots of -1.

For $\xi \in E^{\omega}_{\tau/2}$ and $\eta \in E^{\lambda}_{\tau/2}$ with $\omega \neq \lambda$ are $2m-1$ roots of -1, we have

$$\begin{aligned} Q^{-1}_{(2m-1)\tau/2}(\xi,\ \eta) &= \int_0^{(2m-1)\tau/2} (J\dot{\xi}\cdot\eta + (H''(y(t)))^{-1}J\dot{\xi}\cdot J\dot{\eta})dt \\ &= \sum_{k=0}^{2m-1}(\omega\overline{\lambda})^k \int_0^{\tau/2}(J\dot{\xi}\cdot\eta + (H''(y(t)))^{-1}J\dot{\xi}\cdot J\dot{\eta})dt \\ &= 0. \end{aligned} \tag{3.9}$$

Thus the lemma holds. ∎

Definition 3.3. *We define the Bott maps $j_{\tau/2}$ and $n_{\tau/a}$ from $\mathbf{U}$ to $\mathbf{Z}$ by*

$$j_{\tau/2}(\omega) = \mathrm{index} Q^{\omega}_{\tau/2}, \qquad n_{\tau/2}(\omega) = \mathrm{nullity} Q^{\omega}_{\tau/2}, \tag{3.10}$$

By Lemmas 3.1 and 3.2, we have

Proposition 3.4. *Suppose u^{2m-1} is a nonzero critical point of Ψ_a such that u corresponds to $(\tau, y) \in \mathcal{T}_s(\Sigma)$. Then we have*

$$i(u^{2m-1}) = \sum_{\omega^{2m-1}=-1} j_{\tau/2}(\omega) \qquad \nu(u^{2m-1}) = \sum_{\omega^{2m-1}=-1} n_{\tau/2}(\omega) - 1. \tag{3.11}$$

Note that $j_{\tau/2}(\omega)$ coincide with the function defined in Definition 1.5.3 of [Eke3] for the linear Hamiltonian system

$$\begin{cases} \dot{\xi}(t) = JA(t)\xi \\ A(t+\tau/2) = A(t) \end{cases} \tag{3.12}$$

where $A(t) = H''(y(t))$. Denote by $i^E(A,k)$ and $\nu^E(A,k)$ the index and nullity of the k-th iteration of the system (3.12) defined by Ekeland in [Eke3]. Denote by $i(A,k)$ and $\nu(A,k)$ the Maslov-type

index and nullity of the k-th iteration of the system (3.12) defined by Conley, Zehnder and Long (cf. §5.4 of [Lon4]). Then we have

Theorem 3.5. (cf. Theorem 15.1.1 of [Lon4]) *We have*

$$i^E(A,k) = i(A,k) - n, \quad \nu^E(A,k) = \nu(A,k), \tag{3.13}$$

for any $k \in \mathbf{N}$.

Theorem 3.6. *Suppose* u^{2m-1} *is a nonzero critical point of* Ψ_a *such that* u *corresponds to* $(\tau, y) \in \mathcal{T}_s(\Sigma)$. *Then we have*

$$i(u^{2m-1}) = i_{-1}(A, 2m-1), \qquad \nu(u^{2m-1}) = \nu_{-1}(A, 2m-1) - 1. \tag{3.14}$$

where $i_{-1}(A,k)$ *and* $\nu_{-1}(A,k)$ *are the Maslov-type index and nullity introduced in [Lon2].*

Proof. By Corollary 1.5.4 of [Eke3] and Theorem 9.2.1 of [Lon4] respectively, we have

$$\begin{aligned} i^E(A, 4m-2) &= i^E(A, 2m-1) + i^E_{-1}(A, 2m-1), \\ i(A, 4m-2) &= i(A, 2m-1) + i_{-1}(A, 2m-1) \end{aligned} \tag{3.15}$$

and by Lemma 3.1, we have $i(u^{2m-1}) = i^E_{-1}(A, 2m-1)$. Thus the theorem follows from Theorem 3.5. ∎

Now we compute $i(u^{2m-1})$ via the index iteration method in [Lon4]. First we recall briefly an index theory for symplectic paths. All the details can be found in [Lon4].

As usual, the symplectic group $\mathrm{Sp}(2n)$ is defined by

$$\mathrm{Sp}(2n) = \{M \in \mathrm{GL}(2n, \mathbf{R}) \,|\, M^T J M = J\},$$

whose topology is induced from that of $\mathbf{R}^{4n^2}$. For $\tau > 0$ we are interested in paths in $\mathrm{Sp}(2n)$:

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0,\tau], \mathrm{Sp}(2n)) \,|\, \gamma(0) = I_{2n}\},$$

which is equipped with the topology induced from that of $\mathrm{Sp}(2n)$. The following real function was introduced in [Lon2]:

$$D_\omega(M) = (-1)^{n-1}\overline{\omega}^n \det(M - \omega I_{2n}), \qquad \forall \omega \in \mathbf{U},\ M \in \mathrm{Sp}(2n).$$

Thus for any $\omega \in \mathbf{U}$ the following codimension 1 hypersurface in $\mathrm{Sp}(2n)$ is defined in [Lon2]:

$$\mathrm{Sp}(2n)^0_\omega = \{M \in \mathrm{Sp}(2n) \,|\, D_\omega(M) = 0\}.$$

For any $M \in \mathrm{Sp}(2n)_\omega^0$, we define a co-orientation of $\mathrm{Sp}(2n)_\omega^0$ at M by the positive direction $\frac{d}{dt}Me^{t\epsilon J}|_{t=0}$ of the path $Me^{t\epsilon J}$ with $0 \le t \le 1$ and $\epsilon > 0$ being sufficiently small. Let

$$\begin{aligned}
\mathrm{Sp}(2n)_\omega^* &= \mathrm{Sp}(2n) \setminus \mathrm{Sp}(2n)_\omega^0, \\
\mathcal{P}_{\tau,\omega}^*(2n) &= \{\gamma \in \mathcal{P}_\tau(2n) \,|\, \gamma(\tau) \in \mathrm{Sp}(2n)_\omega^*\}, \\
\mathcal{P}_{\tau,\omega}^0(2n) &= \mathcal{P}_\tau(2n) \setminus \mathcal{P}_{\tau,\omega}^*(2n).
\end{aligned}$$

For any two continuous arcs ξ and $\eta : [0,\tau] \to \mathrm{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, it is defined as usual:

$$\eta * \xi(t) = \begin{cases} \xi(2t), & \text{if } 0 \le t \le \tau/2, \\ \eta(2t-\tau), & \text{if } \tau/2 \le t \le \tau. \end{cases}$$

Given any two $2m_k \times 2m_k$ matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ with $k = 1, 2$, as in [Lon4], the $\diamond$-product of M_1 and M_2 is defined by the following $2(m_1+m_2) \times 2(m_1+m_2)$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Denote by $M^{\diamond k}$ the k-fold $\diamond$-product $M \diamond \cdots \diamond M$. Note that the $\diamond$-product of any two symplectic matrices is symplectic. For any two paths $\gamma_j \in \mathcal{P}_\tau(2n_j)$ with $j = 0$ and 1, let $\gamma_0 \diamond \gamma_1(t) = \gamma_0(t) \diamond \gamma_1(t)$ for all $t \in [0, \tau]$.

A special path $\xi_n \in \mathcal{P}_\tau(2n)$ is defined by

$$\xi_n(t) = \begin{pmatrix} 2 - \frac{t}{\tau} & 0 \\ 0 & (2 - \frac{t}{\tau})^{-1} \end{pmatrix}^{\diamond n} \qquad \text{for } 0 \le t \le \tau. \tag{3.16}$$

Definition 3.7. (cf. [Lon2], [Lon4]) *For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, define*

$$\nu_\omega(M) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I_{2n}). \tag{3.17}$$

For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, define

$$\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)). \tag{3.18}$$

If $\gamma \in \mathcal{P}_{\tau,\omega}^*(2n)$, define

$$i_\omega(\gamma) = [\mathrm{Sp}(2n)_\omega^0 : \gamma * \xi_n], \tag{3.19}$$

where the right hand side of (3.19) is the usual homotopy intersection number, and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed end points.

If $\gamma \in \mathcal{P}^0_{\tau,\omega}(2n)$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of γ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma) = \sup_{U\in\mathcal{F}(\gamma)} \inf\{i_\omega(\beta)\,|\,\beta \in U \cap \mathcal{P}^*_{\tau,\omega}(2n)\}. \tag{3.20}$$

Then

$$(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times \{0, 1, \ldots, 2n\},$$

is called the index function of γ at ω.

For any $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, the *splitting numbers* $S^\pm_M(\omega)$ of M at ω are defined by

$$S^\pm_M(\omega) = \lim_{\epsilon\to 0^+} i_{\omega\exp(\pm\sqrt{-1}\epsilon)}(\gamma) - i_\omega(\gamma), \tag{3.21}$$

for any path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$.

Let $\Omega^0(M)$ be the path connected component containing $M = \gamma(\tau)$ of the set

$$\begin{aligned}\Omega(M) = \{N \in \mathrm{Sp}(2n) \quad | \quad & \sigma(N) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \text{ and} \\ & \nu_\lambda(N) = \nu_\lambda(M)\ \forall\, \lambda \in \sigma(M) \cap \mathbf{U}\}.\end{aligned} \tag{3.22}$$

Here $\Omega^0(M)$ is called the *homotopy component* of M in $\mathrm{Sp}(2n)$.

In [Lon2]-[Lon4], the following symplectic matrices were introduced as *basic normal forms*:

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \lambda = \pm 2, \tag{3.23}$$

$$N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \qquad \lambda = \pm 1, b = \pm 1, 0, \tag{3.24}$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in (0, \pi) \cup (\pi, 2\pi), \tag{3.25}$$

$$N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \qquad \theta \in (0, \pi) \cup (\pi, 2\pi), \tag{3.26}$$

where $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ with $b_i \in \mathbf{R}$ and $b_2 \neq b_3$.

Splitting numbers possess the following properties:

Lemma 3.8. (cf. [Lon2] and Lemma 9.1.5 of [Lon4]) *Splitting numbers $S^\pm_M(\omega)$ are well defined, i.e., they are independent of the choice of the path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$ appeared in (3.21). For $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, splitting numbers $S^\pm_N(\omega)$ are constant for all $N \in \Omega^0(M)$. Moreover, we have*

$$S^\pm_M(\omega) = 0, \qquad \text{if } \omega \notin \sigma(M).$$

$$S^+_M(\omega) = S^-_M(\overline{\omega}), \qquad \forall \omega \in \mathbf{U}.$$

Lemma 3.9. (cf. [Lon2], Lemma 9.1.5 of [Lon4]) *For any $M_i \in \mathrm{Sp}(2n_i)$ with $i = 0$ and 1, there holds*

$$S^{\pm}_{M_0 \diamond M_1}(\omega) = S^{\pm}_{M_0}(\omega) + S^{\pm}_{M_1}(\omega), \qquad \forall\, \omega \in \mathbf{U}. \tag{3.27}$$

We have the following

Theorem 3.10. (cf. [Lon3] and Theorem 1.8.10 of [Lon4]) *For any $M \in \mathrm{Sp}(2n)$, there is a path $f : [0,1] \to \Omega^0(M)$ such that $f(0) = M$ and*

$$f(1) = M_1 \diamond \cdots \diamond M_l, \tag{3.28}$$

where each M_i is a basic normal form listed in (3.23)-(3.26) for $1 \le i \le l$.

Now we deduce the index iteration formula for each case in (3.23)-(3.26), Note that the splitting numbers are computed in List 9.1.12 of [Lon4].

Case 1. *M is conjugate to a matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b > 0$.*

In this case, we have $(S^+_M(1), S^-_M(1)) = (1,1)$. Thus by Theorem 9.2.1 of [Lon4], we have

$$\begin{aligned} i_{-1}(\gamma^{2m-1}) &= \sum_{\omega^{2m-1}=-1} i_\omega(\gamma) = \sum_{k=1}^{2m-1} i_{\frac{(2k-1)\pi}{2m-1}}(\gamma) = (2m-1)(i_1(\gamma)+1), \\ \nu_{-1}(\gamma^{2m-1}) &= 0. \end{aligned} \tag{3.29}$$

Case 2. *$M = I_2$, the 2×2 identity matrix.*

In this case, we have $(S^+_M(1), S^-_M(1)) = (1,1)$. Thus as in Case 1, we have

$$i_{-1}(\gamma^{2m-1}) = (2m-1)(i_1(\gamma)+1), \quad \nu_{-1}(\gamma^{2m-1}) = 0. \tag{3.30}$$

Case 3. *M is conjugate to a matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b < 0$.*

In this case, we have $(S^+_M(1), S^-_M(1)) = (0,0)$. Thus by Theorem 9.2.1 of [Lon4], we have

$$\begin{aligned} i_{-1}(\gamma^{2m-1}) &= \sum_{\omega^{2m-1}=-1} i_\omega(\gamma) = \sum_{k=1}^{2m-1} i_{\frac{(2k-1)\pi}{2m-1}}(\gamma) = (2m-1)i_1(\gamma), \\ \nu_{-1}(\gamma^{2m-1}) &= 0. \end{aligned} \tag{3.31}$$

Case 4. *M is conjugate to a matrix $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ for some $b < 0$.*

In this case, we have $(S^+_M(-1), S^-_M(-1)) = (1,1)$. Thus by Theorem 9.2.1 of [Lon4], we have

$$i_{-1}(\gamma^{2m-1}) = \sum_{\omega^{2m-1}=-1} i_\omega(\gamma) = \sum_{k=1}^{2m-1} i_{\frac{(2k-1)\pi}{2m-1}}(\gamma)$$

$$\begin{aligned}
&= \sum_{k=1}^{m-1} i_{\frac{(2k-1)\pi}{2m-1}}(\gamma) + i_{-1}(\gamma) + \sum_{k=m+1}^{2m-1} i_{\frac{(2k-1)\pi}{2m-1}}(\gamma) \\
&= (m-1)i_1(\gamma) + i_1(\gamma) - 1 + (m-1)(i_1(\gamma) - 1 + 1) \\
&= (2m-1)i_1(\gamma) - 1, \\
\nu_{-1}(\gamma^{2m-1}) &= 1. \qquad (3.32)
\end{aligned}$$

Case 5. $M = -I_2$.

In this case, we have $(S_M^+(-1), S_M^-(-1)) = (1,1)$. Thus as in Case 4, we have

$$i_{-1}(\gamma^{2m-1}) = (2m-1)i_1(\gamma) - 1, \quad \nu_{-1}(\gamma^{2m-1}) = 2. \qquad (3.33)$$

Case 6. *M is conjugate to a matrix* $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ *for some $b > 0$.*

In this case, we have $(S_M^+(-1), S_M^-(-1)) = (0,0)$. Thus by Theorem 9.2.1 of [Lon4], we have

$$\begin{aligned}
i_{-1}(\gamma^{2m-1}) &= \sum_{\omega^{2m-1}=-1} i_\omega(\gamma) = \sum_{k=1}^{2m-1} i_{\frac{(2k-1)\pi}{2m-1}}(\gamma) = (2m-1)i_1(\gamma), \\
\nu_{-1}(\gamma^{2m-1}) &= 1. \qquad (3.34)
\end{aligned}$$

Case 7. $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ *with some $\theta \in (0,\pi) \cup (\pi, 2\pi)$.*

In this case, we have $(S_M^+(e^{\sqrt{-1}\theta}), S_M^-(e^{\sqrt{-1}\theta})) = (0,1)$. Thus by Theorem 9.2.1 of [Lon4] and Lemma 3.8, we have

$$\begin{aligned}
i_{-1}(\gamma^{2m-1}) &= \sum_{\omega^{2m-1}=-1} i_\omega(\gamma) = \sum_{k=1}^{2m-1} i_{\frac{(2k-1)\pi}{2m-1}}(\gamma) \\
&= \sum_{2k-1<\frac{(2m-1)\theta}{\pi}} i_1(\gamma) + \sum_{\frac{(2m-1)\theta}{\pi} \le 2k-1 \le \frac{(2m-1)(2\pi-\theta)}{\pi}} (i_1(\gamma) - 1) \\
&\quad + \sum_{\frac{(2m-1)(2\pi-\theta)}{\pi} < 2k-1 \le 4m-2} i_1(\gamma) \\
&= (2m-1)(i_1(\gamma) - 1) + 2E\left(\frac{(2m-1)\theta}{2\pi} + \frac{1}{2}\right) - 2, \\
\nu_{-1}(\gamma^{2m-1}) &= 2 - 2\phi\left(\frac{(2m-1)\theta}{2\pi} + \frac{1}{2}\right), \qquad (3.35)
\end{aligned}$$

provided $\theta \in (0,\pi)$. When $\theta \in (\pi, 2\pi)$, we have

$$\begin{aligned}
i_{-1}(\gamma^{2m-1}) &= \sum_{\omega^{2m-1}=-1} i_\omega(\gamma) = \sum_{k=1}^{2m-1} i_{\frac{(2k-1)\pi}{2m-1}}(\gamma) \\
&= \sum_{2k-1 \le \frac{(2m-1)(2\pi-\theta)}{\pi}} i_1(\gamma) + \sum_{\frac{(2m-1)(2\pi-\theta)}{\pi} < 2k-1 < \frac{(2m-1)\theta}{\pi}} (i_1(\gamma) + 1)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\frac{(2m-1)\theta}{\pi}\le 2k-1\le 4m-2} i_1(\gamma)\\
&= (2m-1)(i_1(\gamma)-1)+2E\left(\frac{(2m-1)\theta}{2\pi}+\frac{1}{2}\right)-2,\\
\nu_{-1}(\gamma^{2m-1}) &= 2-2\phi\left(\frac{(2m-1)\theta}{2\pi}+\frac{1}{2}\right).
\end{aligned}
$$

Case 8. $M=\begin{pmatrix}R(\theta) & b\\ 0 & R(\theta)\end{pmatrix}$ *with some* $\theta\in(0,\pi)\cup(\pi,2\pi)$ *and* $b=\begin{pmatrix}b_1 & b_2\\ b_3 & b_4\end{pmatrix}\in\mathbf{R}^{2\times 2}$, *such that* $(b_2-b_3)\sin\theta<0$.

In this case, we have $(S_M^+(e^{\sqrt{-1}\theta}),S_M^-(e^{\sqrt{-1}\theta}))=(1,1)$. Thus by Theorem 9.2.1 of [Lon4], we have

$$
\begin{aligned}
i_{-1}(\gamma^{2m-1}) &= \sum_{\omega^{2m-1}=-1} i_\omega(\gamma)=\sum_{k=1}^{2m-1} i_{\frac{(2k-1)\pi}{2m-1}}(\gamma)\\
&= (2m-1)i_1(\gamma)+2\phi\left(\frac{(2m-1)\theta}{2\pi}+\frac{1}{2}\right)-2,\\
\nu_{-1}(\gamma^{2m-1}) &= 2-2\phi\left(\frac{(2m-1)\theta}{2\pi}+\frac{1}{2}\right).
\end{aligned}
\tag{3.36}
$$

Case 9. $M=\begin{pmatrix}R(\theta) & b\\ 0 & R(\theta)\end{pmatrix}$ *with some* $\theta\in(0,\pi)\cup(\pi,2\pi)$ *and* $b=\begin{pmatrix}b_1 & b_2\\ b_3 & b_4\end{pmatrix}\in\mathbf{R}^{2\times 2}$, *such that* $(b_2-b_3)\sin\theta>0$.

In this case, we have $(S_M^+(e^{\sqrt{-1}\theta}),S_M^-(e^{\sqrt{-1}\theta}))=(0,0)$. Thus by Theorem 9.2.1 of [Lon4], we have

$$
\begin{aligned}
i_{-1}(\gamma^{2m-1}) &= \sum_{\omega^{2m-1}=-1} i_\omega(\gamma)=\sum_{k=1}^{2m-1} i_{\frac{(2k-1)\pi}{2m-1}}(\gamma)=(2m-1)i_1(\gamma),\\
\nu_{-1}(\gamma^{2m-1}) &= 2-2\phi\left(\frac{(2m-1)\theta}{2\pi}+\frac{1}{2}\right).
\end{aligned}
\tag{3.37}
$$

Case 10. *M is hyperbolic, i.e.,* $\sigma(M)\cap\mathbf{U}=\emptyset$.

In this case, by Theorem 9.2.1 of [Lon4], we have

$$
\begin{aligned}
i_{-1}(\gamma^{2m-1}) &= \sum_{\omega^{2m-1}=-1} i_\omega(\gamma)=\sum_{k=1}^{2m-1} i_{\frac{(2k-1)\pi}{2m-1}}(\gamma)=(2m-1)i_1(\gamma),\\
\nu_{-1}(\gamma^{2m-1}) &= 0.
\end{aligned}
\tag{3.38}
$$

Proposition 3.11. *For any* $m\in\mathbf{N}$, *we have the estimate*

$$
i_{-1}(\gamma^{2m+1})-i_{-1}(\gamma^{2m-1})-\nu_{-1}(\gamma^{2m-1})\ge 2i_1(\gamma)-e(M). \tag{3.39}
$$

Proof. We consider each of the above cases.

Case 1. *M is conjugate to a matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b > 0$.*

In this case we have

$$i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) - \nu_{-1}(\gamma^{2m-1}) = 2i_1(\gamma) + 2.$$

Case 2. *$M = I_2$, the 2×2 identity matrix.*

In this case we have

$$i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) - \nu_{-1}(\gamma^{2m-1}) = 2i_1(\gamma) + 2.$$

Case 3. *M is conjugate to a matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b < 0$.*

In this case we have

$$i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) - \nu_{-1}(\gamma^{2m-1}) = 2i_1(\gamma).$$

Case 4. *M is conjugate to a matrix $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ for some $b < 0$.*

In this case we have

$$i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) - \nu_{-1}(\gamma^{2m-1}) = 2i_1(\gamma) - 1$$

Case 5. *$M = -I_2$, the 2×2 identity matrix.*

In this case we have

$$i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) - \nu_{-1}(\gamma^{2m-1}) = 2i_1(\gamma) - 2.$$

Case 6. *M is conjugate to a matrix $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ for some $b > 0$.*

In this case we have

$$i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) - \nu_{-1}(\gamma^{2m-1}) = 2i_1(\gamma) - 1.$$

Case 7. $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ *with some $\theta \in (0, \pi) \cup (\pi, 2\pi)$.*

In this case we have

$$\begin{aligned}
& i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) - \nu_{-1}(\gamma^{2m-1}) \\
= \quad & 2(i_1(\gamma) - 1) + 2E\left(\frac{(2m+1)\theta}{2\pi} + \frac{1}{2}\right) - 2E\left(\frac{(2m-1)\theta}{2\pi} + \frac{1}{2}\right) \\
& - \left(2 - 2\phi\left(\frac{(2m-1)\theta}{2\pi} + \frac{1}{2}\right)\right) \\
\geq \quad & 2(i_1(\gamma) - 1).
\end{aligned}$$

Case 8. $M = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}$ *with some* $\theta \in (0, \pi) \cup (\pi, 2\pi)$ *and* $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbf{R}^{2\times 2}$, *such that* $(b_2 - b_3)\sin\theta < 0$.

In this case we have

$$\begin{aligned} & i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) - \nu_{-1}(\gamma^{2m-1}) \\ = \quad & 2i_1(\gamma) + 2\phi\left(\frac{(2m+1)\theta}{2\pi} + \frac{1}{2}\right) - 2\phi\left(\frac{(2m-1)\theta}{2\pi} + \frac{1}{2}\right) \\ & -\left(2 - 2\phi\left(\frac{(2m-1)\theta}{2\pi} + \frac{1}{2}\right)\right) \\ \geq \quad & 2i_1(\gamma) - 2. \end{aligned}$$

Case 9. $M = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}$ *with some* $\theta \in (0, \pi) \cup (\pi, 2\pi)$ *and* $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbf{R}^{2\times 2}$, *such that* $(b_2 - b_3)\sin\theta > 0$.

In this case we have

$$\begin{aligned} & i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) - \nu_{-1}(\gamma^{2m-1}) \\ = \quad & 2i_1(\gamma) - \left(2 - 2\phi\left(\frac{(2m-1)\theta}{2\pi} + \frac{1}{2}\right)\right) \\ \geq \quad & 2i_1(\gamma) - 2. \end{aligned}$$

Case 10. M *is hyperbolic, i.e.,* $\sigma(M) \cap \mathbf{U} = \emptyset$.

In this case we have

$$i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) - \nu_{-1}(\gamma^{2m-1}) = 2i_1(\gamma).$$

Combining the above cases, we obtain the proposition. ∎

4 Proof of the main theorem

In this section, we give the proof of the main theorem. first we have the following.

Lemma 4.1. *Suppose* u^{2k-1} *is a nonzero critical point of* Ψ_a *such that* u *corresponds to* $(\tau, y) \in \mathcal{T}_s(\Sigma)$. *Then we can find* $m \in \mathbf{N}$ *such that*

$$i(u^{2m+1}) - i(u^{2m-1}) \geq 4. \tag{4.1}$$

Proof. Let $(\tau,\, y) \in \mathcal{T}_s(\Sigma)$. The fundamental solution $\gamma_y : [0,\, \tau/2] \to \mathrm{Sp}(2n)$ with $\gamma_y(0) = I_{2n}$ of the linearized Hamiltonian system

$$\dot{w}(t) = JH''(y(t))w(t), \qquad \forall t \in \mathbf{R}, \tag{4.2}$$

is called the *associate symplectic path* of (τ, y). Then as in §1.7 of [Eke3], we have

$$\gamma_y(\tau/2) = \begin{pmatrix} -I_2 & 0 \\ 0 & C \end{pmatrix} \tag{4.3}$$

in an appropriate coordinate. Then by Lemma 3.1 and Theorem 3.5, we have

$$i(u^{2k-1}) = i_{-1}(\gamma^{2k-1}), \quad \nu(u^{2k-1}) = \nu_{-1}(\gamma^{2k-1}), \tag{4.4}$$

for any $k \in \mathbf{N}$. By Theorem 3.10, the matrix $\gamma_y(\tau/2)$ can be connected in $\Omega^0(\gamma_y(\tau/2))$ to a basic form decomposition $M = (-I_2)\diamond M_1\diamond\cdots\diamond M_l$. Since $n \geq 2$, we may write $M = (-I_2)\diamond M_1\diamond M'$, where $M' = M_2\diamond\cdots\diamond M_l$. By the symplectic additivity of indices, cf. [Lon2]-[Lon4], we have

$$i_{-1}(\gamma^{2k-1}) = i_{-1}(\gamma_1^{2k-1}) + i_{-1}(\gamma_2^{2k-1}) \tag{4.5}$$

where γ_1 and γ_2 are appropriate symplectic paths such that $\gamma_1(\tau/2) = (-I_2)\diamond M_1$ and $\gamma_2(\tau/2) = M'$.

Note that by Theorem 3.5, we have $i_1(\gamma) \geq n$. Now we consider each case as in §3.

Case 1. $M_1 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ *for some* $b > 0$ *or* $M_1 = I_2$.

In this case we have

$$\begin{aligned} & i(u^{2m+1}) - i(u^{2m-1}) = i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) \\ = \quad & i_{-1}(\gamma_1^{2m+1}) - i_{-1}(\gamma_1^{2m-1}) + i_{-1}(\gamma_2^{2m+1}) - i_{-1}(\gamma_2^{2m-1}) \\ \geq \quad & 2i_1(\gamma_1) + 2 + 2i_1(\gamma_2) - (2n-4) + \nu_{-1}(\gamma_2^{2m-1}) \\ \geq \quad & 2i_1(\gamma) + 6 - 2n \geq 6. \end{aligned}$$

Note that in the above computations, we use (3.29), (3.30), (3.33), Proposition 3.11 and $i_1(\gamma) \geq n$.

Case 2. *M is conjugate to a matrix* $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ *for some* $b < 0$.

In this case, by (3.31) we have

$$\begin{aligned} & i(u^{2m+1}) - i(u^{2m-1}) = i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) \\ = \quad & i_{-1}(\gamma_1^{2m+1}) - i_{-1}(\gamma_1^{2m-1}) + i_{-1}(\gamma_2^{2m+1}) - i_{-1}(\gamma_2^{2m-1}) \\ \geq \quad & 2i_1(\gamma_1) + 2i_1(\gamma_2) - (2n-4) + \nu_{-1}(\gamma_2^{2m-1}) \\ \geq \quad & 2i_1(\gamma) + 4 - 2n \geq 4. \end{aligned}$$

Case 3. $M = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ *for* $b \in \mathbf{R}$.

In this case, by (3.32)-(3.34) we have

$$
\begin{aligned}
& i(u^{2m+1}) - i(u^{2m-1}) = i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) \\
= \quad & i_{-1}(\gamma_1^{2m+1}) - i_{-1}(\gamma_1^{2m-1}) + i_{-1}(\gamma_2^{2m+1}) - i_{-1}(\gamma_2^{2m-1}) \\
\geq \quad & 2i_1(\gamma_1) + 2i_1(\gamma_2) - (2n-4) + \nu_{-1}(\gamma_2^{2m-1}) \\
\geq \quad & 2i_1(\gamma) + 4 - 2n \geq 4.
\end{aligned}
$$

Case 4. $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ *with some* $\theta \in (0,\pi)\cup(\pi,2\pi)$.

In this case, by (3.35) we have

$$
\begin{aligned}
& i(u^{2m+1}) - i(u^{2m-1}) = i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) \\
= \quad & i_{-1}(\gamma_1^{2m+1}) - i_{-1}(\gamma_1^{2m-1}) + i_{-1}(\gamma_2^{2m+1}) - i_{-1}(\gamma_2^{2m-1}) \\
\geq \quad & 2i_1(\gamma_1) - 2 + 2E\left(\frac{(2m+1)\theta}{2\pi} + \frac{1}{2}\right) - 2E\left(\frac{(2m-1)\theta}{2\pi} + \frac{1}{2}\right) \\
& +2i_1(\gamma_2) - (2n-4) + \nu_{-1}(\gamma_2^{2m-1}) \\
\geq \quad & 2i_1(\gamma) + 4 - 2n \geq 4
\end{aligned}
$$

provided we choose m such that $E\left(\frac{(2m+1)\theta}{2\pi} + \frac{1}{2}\right) - E\left(\frac{(2m-1)\theta}{2\pi} + \frac{1}{2}\right) \geq 1$.

Case 5. $M = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}$ *with some* $\theta \in (0,\pi)\cup(\pi,2\pi)$ *and* $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbf{R}^{2\times 2}$, *such that* $(b_2 - b_3)\sin\theta < 0$.

In this case, by (3.36) we have

$$
\begin{aligned}
& i(u^{2m+1}) - i(u^{2m-1}) = i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) \\
= \quad & i_{-1}(\gamma_1^{2m+1}) - i_{-1}(\gamma_1^{2m-1}) + i_{-1}(\gamma_2^{2m+1}) - i_{-1}(\gamma_2^{2m-1}) \\
\geq \quad & 2i_1(\gamma_1) + 2\varphi\left(\frac{(2m+1)\theta}{2\pi} + \frac{1}{2}\right) - 2\varphi\left(\frac{(2m-1)\theta}{2\pi} + \frac{1}{2}\right) \\
& +2i_1(\gamma_2) - (2n-6) + \nu_{-1}(\gamma_2^{2m-1}) \\
\geq \quad & 2i_1(\gamma) + 4 - 2n \geq 4.
\end{aligned}
$$

Case 6. $M = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}$ *with some* $\theta \in (0,\pi)\cup(\pi,2\pi)$ *and* $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbf{R}^{2\times 2}$, *such that* $(b_2 - b_3)\sin\theta > 0$.

In this case, by (3.37) we have

$$
\begin{aligned}
& i(u^{2m+1}) - i(u^{2m-1}) = i_{-1}(\gamma^{2m+1}) - i_{-1}(\gamma^{2m-1}) \\
= \quad & i_{-1}(\gamma_1^{2m+1}) - i_{-1}(\gamma_1^{2m-1}) + i_{-1}(\gamma_2^{2m+1}) - i_{-1}(\gamma_2^{2m-1}) \\
\geq \quad & 2i_1(\gamma_1) + 2i_1(\gamma_2) - (2n-6) + \nu_{-1}(\gamma_2^{2m-1}) \\
\geq \quad & 2i_1(\gamma) + 6 - 2n \geq 6.
\end{aligned}
$$

Case 7. *M is hyperbolic, i.e., $\sigma(M)\cap \mathbf{U}=\emptyset$.*

In this case, by (3.38) we have

$$\begin{aligned}
& i(u^{2m+1})-i(u^{2m-1})=i_{-1}(\gamma^{2m+1})-i_{-1}(\gamma^{2m-1})\\
= \quad & i_{-1}(\gamma_1^{2m+1})-i_{-1}(\gamma_1^{2m-1})+i_{-1}(\gamma_2^{2m+1})-i_{-1}(\gamma_2^{2m-1})\\
\geq \quad & 2i_1(\gamma_1)+2i_1(\gamma_2)-(2n-4)+\nu_{-1}(\gamma_2^{2m-1})\\
\geq \quad & 2i_1(\gamma)+4-2n\geq 4.
\end{aligned}$$

Combining all the above cases, we obtain the lemma. ∎

Proof of Theorem 1.1. We prove by contraction. Assume $\mathcal{T}_s(\Sigma)=\{(\tau,y)\}$. Suppose u^{2m-1} is a nonzero critical point of Ψ_a such that u corresponds to $(\tau,y)\in\mathcal{T}_s(\Sigma)$. By Lemma 4.1, we may assume $i(u^{2m+1})-i(u^{2m-1})\geq 4$. The index interval of (τ,y) at $2m-1$ is defined to be $\mathcal{G}_{2m-1}=(i(u^{2m-3})+\nu(u^{2m-3})-1,\ i(u^{2m+1}))$. Note that by Proposition 3.11 and $i_1(\gamma)\geq n$, we have $i(u^{2m-3})+\nu(u^{2m-3})\leq i(u^{2m-1})$. Thus we have $(i(u^{2m-1})-1,\ i(u^{2m+1}))\subset\mathcal{G}_{2m-1}$. Hence we can find two distinct even integers $2T_1,2T_2\in\mathcal{G}_{2m-1}$. Let c_{T_1+1} and c_{T_2+1} be the two critical values of Ψ_a found by Proposition 2.16. Then we have $c_{T_1+1}\neq c_{T_2+1}$ since ${}^{\#}\mathcal{T}_s(\Sigma)<\infty$. By Proposition 2.17, we have

$$\begin{aligned}
\Psi_a(u^{2m_1-1})=c_{T_1+1}, &\qquad i(u^{2m_1})\leq 2T_1\leq i(u^{2m_1-1})+\nu(u^{2m_1-1})-1,\\
\Psi_a(u^{2m_2-1})=c_{T_2+1}, &\qquad i(u^{2m_2})\leq 2T_2\leq i(u^{2m_2-1})+\nu(u^{2m_2-1})-1,
\end{aligned}\tag{4.6}$$

for some $m_1,m_2\in\mathbf{N}$. On the other hand, we must have $m_1=m_2$ by Proposition 3.11. Thus we have $c_{T_1+1}=c_{T_2+1}$. This contradiction proves the theorem. ∎

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